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DYNAMIC STABILITY OF A CANTILEVER BEAM ATTACHED TO A TRANSLATIONAL/ ROTATIONAL BASE

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The dynamic stability of a cantilever beam attached to a translational/ rotational base is studied in this paper. Equations of motion for the simple flexure cantilever beam with a tip mass are derived by Hamilton's principle, and then transformed into a set of ordinary differential equations by applying variable transformation and the Galerkin method. Hsu's method is extended to investigate the instability regions of the non-homogeneous solutions. The main objective of this paper is to identify instability regions of the system for various combinations of the excitation frequencies and amplitudes of the oscillations. The instability regions of the system with and without tip mass and effects of the rotational angle velocities are compared and discussed by using Hsu's and Bolotin's methods.

1. INTRODUCTION

The dynamic problems of a cantilever beam attached to a moving base are associated with various applications such as elastic linkages, rotating machinery, robot manipulator arms, aircraft propellers, helicopter rotor blades, flexible satellites, the textile industry, and flexible appendages of a spacecraft. The dynamic stability of ordinary differential equations with periodic coefficients was studied using Hsu's method [1], wherein a first approximation analysis was carried out and criteria for instability were derived. Most studies only considered lateral deflection, and homogeneous solutions are solved using Hsu's method. Elmaraghy and Tabarrok [2] employed both Hsu's and Bolotin's [3] methods to investigate the dynamic stability of an axially oscillating Euler beam.

Numerous studies have used different theories and techniques to investigate the dynamic stability of belts and chains in mechanical machinery [4–6]. Tsuchiya [7] analyzed the attitude behavior of a spacecraft with a rotor during extension of flexible appendages. Wang and Wei [8] studied a flexible robot arm as a moving slender prismatic beam. Kane *et al*. [9] investigated a Timoshenko beam built into a rigid base undergoing general three-dimensional motion.

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In order to obtain the deployment responses of a flexible beam, Creamer [10] presented a model using Timoshenko beam theory in conjunction with base oscillatory motion. Yuh and Young [11] derived a time-varying partial differential equation and boundary conditions for an axially moving beam with rotation. Tadikonda and Baruh [12] presented a complete dynamic model for a translating flexible beam with a prismatic joint. Stylianou and Tabarrok [13, 14] solved an axially moving beam problem by using finite element method in which elements are functions of time. Lee [15] exploited the properties of eigenfunctions of a uniform fixed–free beam. The equations of motion were formulated in matrix form for the dynamic responses of an orthotropic rotating shaft moving longitudinally over a spring support.

In this paper, Hamilton's principle is appled to derive the governing equations of a cantilever beam attached to a translational/rotational base. The variable transformation and Galerkin method are employed to discretize the distributed system to a set of ordinary differential equations. In this study, Hsu's method is extended to solve the non-homogeneous problems. The main objective of this paper is to identify the regions of instability for various combinations of the excitation frequencies and amplitudes of the oscillations using both Hsu's and Bolotin's methods.

2. EQUATION FORMULATION

In this section, Hamilton's principle is employed to derive the governing equations of a cantilever beam attached to a translational/rotational base, which is shown in Figure 1. A point mass m_e is attached at the tip end of the cantilever beam. Material properties of the beam are length ℓ, mass density ρ, flexural rigidity EI and uniform cross-section area A_e. The beam is attached to a rigid base which moves translationally and rotationally in the XY-plane. The co-ordinate system OXY is a fixed inertia one. The moving co-ordinate system oxy

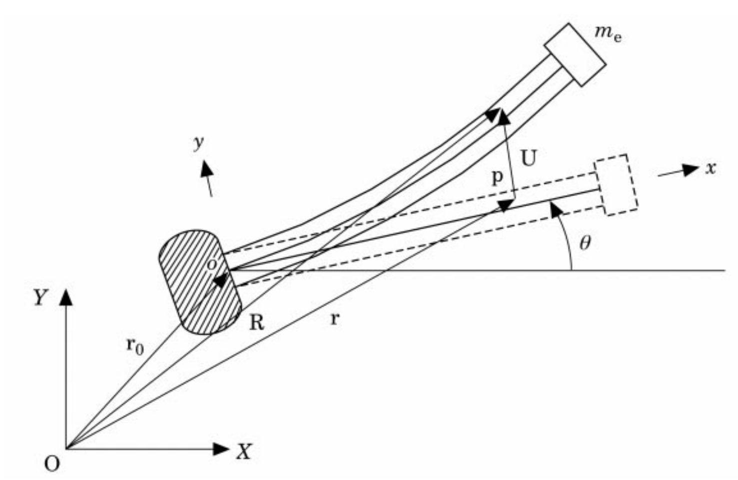

Figure 1. Schematic of a cantilever beam attached to translational/rotational base.

is attached to the rigid base which has the translational/rotational motion. The origin o is located at the rotating point and its potion vector $\mathbf{r}_0$ is $(a(t), b(t))$ and is measured from the fixed co-ordinate. In Appendix A, Timoshenko beam theory is used to derive the governing equations. A reduction process of the system equations through various theories is also presented.

2.1. SIMPLE FLEXURE MODEL

The simple flexure model assumes that the axial deformation can be ignored but the inertia effect of the translational and rotational motions of the base is retained. The geometric stiffening effects caused by the coupling between the axial and transverse deformations are not included. From Appendix A one has the resultant equation of motion

$$\begin{aligned} EIv_{xxxx} + \rho A v_{tt} - \rho A \Bigg\{ \dot{\theta}^2 [v - x v_x + \tfrac{1}{2}(\ell^2 - x^2) v_{xx}] - 2\dot{\theta}\left(v_t v_x - \int_x^\ell v_t v_{xx}\, \mathrm{d}x \right) \\ - \ddot{\theta}\left(x + v v_x - \int_x^\ell v v_{xx}\, \mathrm{d}x \right) + [-b_{tt} + a_{tt}(v_x - (\ell - x) v_{xx})] \cos\theta \\ + [a_{tt} + b_{tt}(v_x - (\ell - x) v_{xx})] \sin\theta \Bigg\} \\ + m_e v_{xx} [a_{tt} \cos\theta + b_{tt} \sin\theta - 2\dot{\theta} v_t(\ell, t) - \ddot{\theta} v(\ell, t) - \dot{\theta}^2 \ell] = 0, \end{aligned} \tag{1}$$

and the boundary conditions

$$v(0, t) = v_x(0, t) = v_{xx}(\ell, t) = 0, \tag{2a–c}$$

$$\begin{aligned} EIv_{xxx}(\ell, t) - m_e[v_{tt}(\ell, t) + \ddot{\theta}(\ell + v(\ell, t) v_x(\ell, t)) + \dot{\theta}^2 (v_x \ell - v(\ell, t)) \\ + 2\dot{\theta} v_x v_t + (b_{tt} - v_x a_{tt}) \cos\theta - (a_{tt} + v_x b_{tt}) \sin\theta] = 0. \end{aligned} \tag{2d}$$

Equation (2d) is the dynamic equilibrium equation for the tip mass, which includes elastic shear force and inertia force due to motions of both translation and rotation of the base.

The merit of the simple flexure model is that only one governing equation will be solved and the effect of translational/rotational base is still retained in the governing equation (1) and boundary condition (2d). However, the boundary condition (2d) is non-homogeneous, and a special variable transformation is required before the Galerkin method is applied.

2.2. VARIABLE TRANSFORMATION

First, one introduces some non-dimensional quantities as follows:

$$\begin{aligned} V = v/\ell, \quad \tau = \omega_T t, \quad \xi = x/\ell, \quad \Theta = \dot{\theta}/\omega_T, \quad A(\tau) = a(t)/\ell, \\ B(\tau) = b(t)/\ell, \quad \bar{m}_e = m_e/\rho A_e \ell, \end{aligned} \tag{3}$$

where $\omega_T^2 = EI/\rho A_e \ell^4$.

Substituting (3) into equation (1) and assuming $\dot{\theta}$ is constant, one has the dimensionless governing equation

$$V_{\xi\xi\xi\xi} + V_{\tau\tau} - [\Theta^2(V - \xi V_\xi + \tfrac{1}{2}(1-\xi^2)V_{\xi\xi})] + 2\Theta\left[V_\tau V_\xi - \int_\xi^1 V_\tau V_{\xi\xi}\,\mathrm{d}\xi\right]$$
$$- \{-B_{\tau\tau} + A_{\tau\tau}[V_\xi - (1-\xi)V_{\xi\xi}]\}\cos\Theta\tau$$
$$- \{A_{\tau\tau} + B_{\tau\tau}[V_\xi - (1-\xi)V_{\xi\xi}]\}\sin\Theta\tau$$
$$+ \bar{m}_e V_{\xi\xi}(A_{\tau\tau}\cos\Theta\tau + B_{\tau\tau}\sin\Theta\tau - 2\Theta V_\tau - \Theta^2) = 0, \tag{4}$$

and the boundary conditions become

$$V_{\xi\xi\xi}(1,\tau) - \bar{m}_e\{V_{\tau\tau}(1,\tau) + 2\Theta V_\xi(1,\tau)V_\tau(1,\tau) + \Theta^2[V_\xi(1,\tau) - V(1,\tau)]$$
$$+ [B_{\tau\tau} - V_\xi(1,\tau)A_{\tau\tau}]\cos\Theta\tau - [A_{\tau\tau} + V_\xi(1,\tau)B_{\tau\tau}]\sin\Theta\tau\} = 0, \tag{5}$$

$$V(0,\tau) = V_\xi(0,\tau) = V_{\xi\xi}(1,\tau) = 0. \tag{6a–c}$$

In order to apply the Galerkin method, it is necessary to simplify the non-homogeneous boundary condition (5) by using the following variable transformation [16] as

$$V(\xi,\tau) = \bar{V}(\xi,\tau) + F(\xi)h(\tau). \tag{7}$$

where $F(\xi) = \frac{1}{24}\xi^4 - \frac{1}{2}\xi^2$ and $h(\tau) = V_{\xi\xi\xi}(1,\tau)$. Substituting equation (7) into equations (4), (5) and (6a, b, c), one obtains the following equation of motion

$$\bar{V}_{\xi\xi\xi\xi} + \bar{m}_e(A_{\tau\tau}\cos\Theta\tau + B_{\tau\tau}\sin\Theta\tau + 2\Theta\bar{V}_\tau + 2\Theta Fh_\tau - \Theta^2)\bar{V}_{\xi\xi}$$
$$+ (\Theta^2 - B_{\tau\tau}\sin\Theta\tau - A_{\tau\tau}\cos\Theta\tau)\bar{V}_\xi + \bar{V}_{\tau\tau} - \Theta^2\bar{V} + h_{\tau\tau}F$$
$$+ B_{\tau\tau}\cos\Theta\tau - A_{\tau\tau}\sin\Theta\tau$$
$$+ h[-\Theta^2(F - F_\xi) + 1 - A_{\tau\tau}F_\xi\cos\Theta\tau - B_{\tau\tau}F_\xi\sin\Theta\tau]$$
$$+ \bar{m}_e F_{\xi\xi}(A_{\tau\tau}\cos\Theta\tau + B_{\tau\tau}\sin\Theta\tau + 2\Theta\bar{V}_\tau + 2\Theta Fh_\tau - \Theta^2)$$
$$+ 2\Theta(\bar{V}_\tau\bar{V}_\xi + hF_\xi\bar{V}_\tau + h_\tau F\bar{V}_\xi + hh_\tau FF_\xi) = 0, \tag{8}$$

and the homogeneous boundary conditions

$$\bar{V}(0,\tau) = \bar{V}_\xi(0,\tau) = \bar{V}_{\xi\xi}(1,\tau) = \bar{V}_{\xi\xi\xi}(1,\tau) = 0, \tag{9a–d}$$

thus equation (5) becomes

$$\bar{m}_e\bar{V}_{\tau\tau}(1,\tau) + \bar{m}_e Fh_{\tau\tau} - \bar{m}_e(A_{\tau\tau}\cos\Theta\tau + B_{\tau\tau}\sin\Theta\tau - \Theta^2)\bar{V}_\xi(1,\tau) - \bar{m}_e\Theta^2\bar{V}(1,\tau)$$
$$- h[F_{\xi\xi\xi} + \bar{m}_e\Theta^2(F - F_\xi) + \bar{m}_e F_\xi(A_{\tau\tau}\cos\Theta\tau + B_{\tau\tau}\sin\Theta\tau)]$$
$$+ \bar{m}_e(B_{\tau\tau}\cos\Theta\tau - A_{\tau\tau}\sin\Theta\tau)$$
$$+ 2\Theta\bar{m}_e[\bar{V}_\xi(1,\tau)\bar{V}_\tau(1,\tau) + h_{\tau\tau}F\bar{V}_\xi(1,\tau) + hF_\xi\bar{V}_\tau(1,\tau) + hh_\tau FF_\xi] = 0. \tag{10}$$

After the variable transformation (7), the non-homogeneous boundary condition

(5) is changed to homogeneous boundary condition (9) and one additional equation (10).

3. THE DISCRETIZED EQUATION

The governing equation (8) of the cantilever beam attached to a translation/rotational base does not lend itself to a closed-form solution. To obtain the approximate solutions, the displacement $\bar{V}(\xi, \tau)$ can be expressed as a series in terms of a given shape function $\phi_i(\xi)$ with an undetermined coefficient $f_i(\tau)$, where $\phi_i(\xi)$ satisfies the kinematical boundary conditions (9a–d). To obtain the approximate solutions, n terms of the assumed modes are used to expand the continuous displacement field $\bar{V}(\xi, \tau)$ as

$$\bar{V}(\xi, \tau) = \sum_{i=1}^{n} \phi_i(\xi) f_i(\tau). \tag{11}$$

To obtain the instability regions by applying both Hsu's and Bolotin's methods, by substituting equation (11) into equation (8) and applying the Galerkin method in the linearized system, one obtains the governing equation

$$\ddot{f}_i + \omega_{ie}^2 f_i = -\left[\left(\bar{m}_e \sum_{j=1}^{n} N_{ij} - \sum_{j=1}^{n} Q_{ij}\right)(A_{\tau\tau}\cos\Theta\tau + B_{\tau\tau}\sin\Theta\tau)\right] f_i$$
$$+ B_{\tau\tau}\cos\Theta\tau - A_{\tau\tau}\sin\Theta\tau + z, \tag{12}$$

where

$$\omega_{ie}^2 = \Theta^2(Q_{ii} - \bar{m}_e N_{ii} - D_{ii}) + M_{ii},$$

$$z = -h_{\tau\tau}F - h[-\Theta^2(F - F_\xi) + 1 - A_{\tau\tau}F_\xi\cos\Theta\tau - B_{\tau\tau}F_\xi\sin\Theta\tau$$
$$+ \bar{m}_e F_{\xi\xi}(A_{\tau\tau}\cos\Theta\tau + B_{\tau\tau}\sin\Theta\tau - \Theta^2)],$$

and Q_{ii}, N_{ij}, D_{ij} and M_{ij} are shown in Appendix B.

4. STABILITY ANALYSIS BY HSU'S METHOD

Hsu's method [1] is a special perturbation method and combines the method of variation of parameters and the series expansion of the perturbation method. The origin position, $(A(\tau), B(\tau))$, of the translational/rotational base is assumed to be a small perturbation parameter as follows:

$$A(\tau) = \varepsilon\cos\bar{\Theta}\tau, \quad B(\tau) = \varepsilon\sin\bar{\Theta}\tau, \tag{13a, b}$$

where $\bar{\Theta}$ is the frequency of translational motion in the X and Y directions and $\varepsilon > 0$ is a small parameter.

By substituting equation (13a, b) into equation (12), and using trigonometric identities, one gets a different equation with small periodic perturbations as:

TABLE 1
Results from application of Hsu's method

	$\omega_{1e} \cong \frac{1}{2}(\bar{\Theta} - \Theta)$	$\omega_{1e} + \omega_{2e} \approx \bar{\Theta} - \Theta$	$\omega_{2e} - \omega_{1e} \approx \bar{\Theta} - \Theta$
Unstable	$\lambda_{1e}^2 < \alpha$	$\lambda_{2e}^2 < \beta$	$\lambda_{3e}^2 < \gamma$
Neutrally stable	$\lambda_{1e}^2 = \alpha$	$\lambda_{2e}^2 = \beta$	$\lambda_{3e}^2 = \gamma$
A_{iq} and B_{iq} are periodic	$\lambda_{1e}^2 > \alpha$	$\lambda_{2e}^2 > \beta$	$\lambda_{3e}^2 > \gamma$

$$\frac{d^2 f_i}{d\tau^2} + \omega_{ie}^2 f_i = -\varepsilon\bar{\Theta}^2\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n} Q_{ij} - \bar{m}_e \sum_{j=1}^{n} N_{ij}\right)\cos(\bar{\Theta} - \Theta)\tau\right] f_i + \varepsilon\bar{\Theta}^2 \sin(\bar{\Theta} - \Theta)\tau + z. \tag{14}$$

As $\varepsilon = 0$, the non-homogeneous solution of equation (14) is

$$f_i(\tau) = A_{iq}(\tau)\cos\omega_{ie}\tau + B_{iq}(\tau)\sin\omega_{ie}\tau + z/\omega_{ie}^2, \tag{15a}$$

and the solution for $\varepsilon > 0$ in a first order approximation is assumed to be of the form

$$f_i(\tau) = A_{iq}(\tau)\cos\omega_{ie}\tau + B_{iq}(\tau)\sin\omega_{ie}\tau + z/\omega_{ie}^2 + \varepsilon f_{ip}^{(1)}(\tau), \tag{15b}$$

where $A_{iq}(\tau)$ and $B_{iq}(\tau)$ are the slowly varying functions of time.

From Hsu's method [1], the particular integral in (15b) is obtained as

$$\begin{aligned} f_{ip}^{(1)}(\tau) = -\frac{1}{2}\sum_{i,j=1}^{n}\Bigg\{ &\frac{1}{\omega_{ie}^2 - (\bar{\Theta} - \Theta + \omega_{je})^2}[T_{ij}\cos(\bar{\Theta} - \Theta + \omega_{je})\tau \\ &+ V_{ij}\sin(\bar{\Theta} - \Theta - \omega_{je})\tau] \\ &+ \frac{1}{\omega_{ie}^2 - (\bar{\Theta} - \Theta - \omega_{je})^2}[U_{ij}\cos(\bar{\Theta} - \Theta - \omega_{je})\tau + W_{ij}\sin(\bar{\Theta} - \Theta - \omega_{je})\tau] \\ &- 2\left[\frac{1}{\omega_{ie}^2 - (\bar{\Theta} - \Theta)^2}\right][E_{ij}\cos(\bar{\Theta} - \Theta)\tau + F_{ij}\sin(\bar{\Theta} - \Theta)\tau]\Bigg\}, \end{aligned} \tag{16}$$

where T_{ij}, V_{ij}, U_{ij}, W_{ij}, E_{ij} and F_{ij} are functions of A_i and B_i. These coefficients depend upon the physical properties of the system and are defined in Appendix B.

4.1. STABILITY ANALYSIS OF BEAM WITH TIP MASS

For the homogeneous solutions, E_{ij} and F_{ij} must be deleted in equation (16). Three cases using Hsu's method will be discussed: (1) ω_{1e} is near $\frac{1}{2}(\bar{\Theta} - \Theta)$. In this case the denominator $\omega_{1e}^2 - (\bar{\Theta} - \Theta - \omega_{je})^2$ approaches zero in equation

(16), a solution is sought for $2\omega_{1e} + \varepsilon\lambda_{1e} = \bar{\Theta} - \Theta$, and λ_{1e} is a finite real number (2) $\omega_{1e} + \omega_{2e}$ is near $\bar{\Theta} - \Theta$. In this case the denominators $\omega_{1e}^2 - (\bar{\Theta} - \Theta - \omega_{je})^2$ of the equation (16) approach zero as $\omega_{je} = \omega_{2e}$ and $\omega_2 - (\bar{\Theta} - \Theta - \omega_{je})^2$ approach zero as $\omega_{je} = \omega_{1e}$, a solution is sought for $\bar{\Theta} - \Theta = \omega_{1e} + \omega_{2e} + \varepsilon\lambda_{2e}$, and the case is known as "combination resonance of the sum type". (3) $\omega_{2e} - \omega_{1e}$ is near $\bar{\Theta} - \Theta$. In this case the denominators $\omega_{1e}^2 - (\bar{\Theta} - \Theta - \omega_{je})^2$ of equation (16) approach zero as $\omega_{je} = \omega_{2e}$, and $\omega_2^2 - (\bar{\Theta} - \Theta + \omega_{je})^2$ approach zero as $\omega_{je} = \omega_{1e}$, and a solution is sought for $\bar{\Theta} - \Theta = \omega_{2e} - \omega_{1e} + \varepsilon\lambda_{3e}$. This case is called "combination resonance of the difference type". The following results are found by using Hsu's method:

$$\alpha = k_{11}^2/4\omega_{1e}^2, \quad \beta = (k_{12})(k_{21})/4\omega_{1e}\omega_{2e}, \quad \gamma = -(k_{21})(k_{12})/4\omega_{1e}\omega_{2e}.$$

In this paper, Hsu's method is extended to investigate the non-homogeneous solutions of equation (14). The derivative can be seen in Appendix C. The denominator $\omega_{1e}^2 - (\bar{\Theta} - \Theta)^2$ is nearly equal to zero, and a solution is found for $\omega_{1e} + \varepsilon\lambda = \bar{\Theta} - \Theta$ where λ is a finite real number. Thus, one has

$$\begin{aligned} B_{1q} = &-\frac{\varepsilon}{\omega_{1e}} E_{1j} \left\{ \frac{\sin[2(\bar{\Theta} - \Theta) - \varepsilon\lambda]\tau}{2(\bar{\Theta} - \Theta) - \varepsilon\lambda} + \frac{\sin\varepsilon\lambda\tau}{\varepsilon\lambda} \right\} \\ &+ \frac{\varepsilon}{\omega_{1e}} F_{1j} \left\{ \frac{\cos[2(\bar{\Theta} - \Theta) - \varepsilon\lambda]\tau}{2(\bar{\Theta} - \Theta) - \varepsilon\lambda} + \frac{\cos\varepsilon\lambda\tau}{\varepsilon\lambda} \right\} \\ &- \frac{z}{\omega_{1e}} \left(\frac{1 - \omega_{1e}^2}{\omega_{1e}^2} \right) \left[\frac{\sin(\bar{\Theta} - \Theta - \varepsilon\lambda)\tau}{\bar{\Theta} - \Theta - \varepsilon\lambda} \right] + c, \end{aligned} \tag{17}$$

where c is a constant. From equation (17), B_{1q} is unstable only for $\lambda = 0$ or $\varepsilon = 0$. Similarly, one has

$$\begin{aligned} A_{1q} = &\frac{\varepsilon}{\omega_{1e}} E_{1j} \left\{ -\frac{\cos[2(\bar{\Theta} - \Theta) - \varepsilon\lambda]\tau}{2(\bar{\Theta} - \Theta) - \varepsilon\lambda} + \frac{\cos(\varepsilon\lambda\tau)}{\varepsilon\lambda} \right\} \\ &+ \frac{\varepsilon}{\omega_{1e}} F_{1j} \left\{ -\frac{\sin[2(\bar{\Theta} - \Theta) - \varepsilon\lambda]\tau}{2(\bar{\Theta} - \Theta) - \varepsilon\lambda} + \frac{\sin(\varepsilon\lambda\tau)}{\varepsilon\lambda} \right\} \\ &- \frac{z}{\omega_{1e}} \left(\frac{1 - \omega_{1e}^2}{\omega_{1e}^2} \right) \left(\frac{\cos\omega_{1e}\tau}{\bar{\Theta} - \Theta - \varepsilon\lambda} \right) + c_1, \end{aligned} \tag{18}$$

where c_1 is a constant and A_{1q} is unstable only for $\lambda = 0$ or $\varepsilon = 0$. Following similar procedures, equations for A_{2q} and B_{2q} are:

$$A_{2q} = -\left(\frac{z}{\omega_{2e}}\right)\left(\frac{1 - \omega_{2e}^2}{\omega_{2e}^2}\right)\frac{\cos\omega_{2e}\tau}{\omega_{2e}}, \quad B_{2q} = -\left(\frac{z}{\omega_{2e}}\right)\left(\frac{1 - \omega_{2e}^2}{\omega_{2e}^2}\right)\frac{\sin\omega_{2e}\tau}{\omega_{2e}}. \tag{19, 20}$$

4.2. STABILITY ANALYSIS FOR THE BEAM WITHOUT TIP MASS

As the system is without the tip mass, $m_e = 0$ is substituted into the governing equation (1) and boundary conditions. Equation (12) can be simplified as:

$$\ddot{f}_i + \omega_{ig}^2 f_i = -\varepsilon[\bar{\Theta}^2[Q]\cos(\bar{\Theta} - \Theta)\tau]f_i + \varepsilon\bar{\Theta}^2 \sin(\bar{\Theta} - \Theta)\tau, \quad (21)$$

where $\omega_{ig}^2 = M_{ii} + \bar{\Theta}^2(Q_{ii} - D_{ii})$.

The homogeneous solution of equation (21) is substituted by using Hsu's method [1]. Three cases similar to section 4.1 are as follows: (1) $\bar{\Theta} - \Theta = 2\omega_{1g} + \varepsilon\lambda_{1g}$; condition for the instability regions of the system is $\lambda_{1g}^2 < d_{11}^2/4\omega_{1g}^2$. (2) $\bar{\Theta} - \Theta = \omega_{1g} + \omega_{2g} + \varepsilon\lambda_{2g}$; condition for the instability regions is $\lambda_{2g}^2 < (d_{12})(d_{21})/4\omega_{1g}\omega_{2g}$. (3) $\bar{\Theta} - \Theta = \omega_{2g} - \omega_{1g} + \varepsilon\lambda_{3g}$; condition for the instability regions is $\lambda_{3g}^2 < -(d_{12})(d_{21})/4\omega_{1g}\omega_{2g}$, where $d_{ij} = \sum_{j=1}^{n} Q_{ij}$.

The non-homogeneous solutions are solved next via a similar method as set out in Appendix C. One gets $A_{2g} = A_{20}$, $B_{2g} = B_{20}$, where A_{20} and B_{20} are constants and the stability criterion depends on A_{1g} and B_{1g}. If $\omega_{1g} + \varepsilon\lambda_g = \bar{\Theta} - \Theta$ or $\varepsilon = \lambda_g = 0$, one obtains

$$A_{1g} = (\varepsilon\phi'/2\omega_{1g})[\tau - \sin(2\omega_{1g})\tau/2\omega_{1g}] + C_{1g}, \quad (22)$$

$$B_{1g} = (\varepsilon\phi'/2\omega_{1g})\cos(2\omega_{1g})\tau + C_{2g}, \quad (23)$$

where C_{1g} and C_{2g} are constants. When $\varepsilon \neq 0$ or $\lambda_g \neq 0$,

$$A_{1g} = (\varepsilon\phi'/2\omega_{1g})[\sin\varepsilon\lambda_g\tau/\varepsilon\lambda_g - \sin(2\omega_{1g} - \varepsilon\lambda_g)\tau/(2\omega_{1g} - \varepsilon\lambda_g)] + C_{1g}, \quad (24)$$

$$B_{1g} = -(\varepsilon\phi'/2\omega_{1g})[-\cos(2\omega_{1g} + \varepsilon\lambda_g)\tau/(2\omega_{1g} + \varepsilon\lambda_g) - \cos\varepsilon\lambda_g\tau/\varepsilon\lambda_g] + C_{2g}. \quad (25)$$

From equations (24) and (25), when $\lambda_g = 0$ or $\omega_{1g} = \bar{\Theta} - \Theta$, A_{1g} is unstable but B_{1g} is stable as τ increases.

4.3. AXIAL TRANSLATIONS ONLY

As the system is without the tip mass ($m_e = 0$) and only has the axial translations in the x direction ($\theta = \dot{\theta} = \ddot{\theta} = 0$), the governing equation (1) of the simple flexure beam can be reduced to

$$EIV_{xxxx} + \rho AV_{tt} - \rho A\{-b_{tt} + a_{tt}[V_x - (L - x)V_{xx}]\} = 0, \quad (26)$$

and boundary conditions are

$$V(0, t) = V_x(0, t) = V_{xx}(L, t) = V_{xxx}(L, t) = 0. \quad (27)$$

Then applying the Galerkin method, one obtains

$$\ddot{f}_j(\tau) + \left[\omega_{jp}^2 + \varepsilon\bar{\Theta}^2 \cos\bar{\Theta}\tau\left(\sum_{i=1}^{n} P_{ij}\right)\right]f_j(\tau) = 0, \quad j = 1, 2, \ldots, n, \quad (28)$$

where $\omega_{jp}^2 = M_{jj}$ and $P_{ij} = -Q_{ij} + N_{ij} - R_{ij}$. From Hsu's method [1], it is seen that only P_{ij} have the effect on the instability regions of the system. The unstable regions will be obtained as follows:

$$\omega_{1p} \text{ is near } 1/2, \text{ the system is unstable if } \lambda_{1p}^2 < \bar{\Theta}^4 P_{11}^2 / 4\omega_{1p}^2, \tag{29}$$

$$\omega_{1p} + \omega_{2p} \text{ is near 1, the system is unstable if } \lambda_{2p}^2 < \bar{\Theta}^4 P_{12} P_{21} / 4\omega_{1p}\omega_{2p}, \tag{30}$$

$$\omega_{2p} - \omega_{1p} \text{ is near 1, the system is unstable if } \lambda_{3p}^2 < -\bar{\Theta}^4 P_{12} P_{21} / 4\omega_{1p}\omega_{2p}, \tag{31}$$

where $2\omega_{1p} + \varepsilon\lambda_{1p} = 1$, $\omega_{1p} + \omega_{2p} + \varepsilon\lambda_{2p} = 1$ and $\omega_{2p} - \omega_{1p} + \varepsilon\lambda_{3p} = 1$.

5. STABILITY BOUNDARIES BY BOLOTIN'S METHOD

Bolotin's method finds the boundaries of instability regions in the position parameter ε vs. excitation frequency $\bar{\Theta}$ by virtue of the existence of a periodic solution with periods T and $2T$. Thus, the solutions on these boundaries can be represented in Fourier series form

$$\{f\} = \{a\} + \sum_{k=1}^{\infty} (c_k \cos k\bar{\Theta}\tau + d_k \sin k\bar{\Theta}\tau + c_{k/2} \cos \tfrac{1}{2}k\bar{\Theta}\tau + d_{k/2} \sin \tfrac{1}{2}k\bar{\Theta}\tau), \tag{32}$$

where a, c_k, d_k, $c_{k/2}$, and $d_{k/2}$ are constants.

In the following cases we will discuss: (1) a cantilever beam with a tip mass $m_e \neq 0$, (2) axial translation only.

Case 1: substituting equation (32) into equation (14), the principal regions of instability are obtained from the zeros of the central elements of the system as

$$\begin{bmatrix} -(\phi^2/4)[I] + [M_1] - (\varepsilon/2)[D_1] & 0 \\ 0 & -(\phi^2/4)[I] + [M_1] + (\varepsilon/2)[D_1] \end{bmatrix} = 0, \tag{33}$$

where $\phi = \bar{\Theta} - \Theta$, $[M_1] = \omega_{ie}^2$ and $[D_1] = \bar{\Theta}^2([Q] - \bar{m}_e[N])$. These principal regions of instability are composed of single sine and cosine harmonics.

Case 2: equation (28) can be cast in the following first order form:

$$\{\dot{X}(\tau)\} = [H(\tau)]\{X(\tau)\}, \tag{34}$$

where

$$\{\dot{X}(\tau)\} = \begin{bmatrix} f(\tau) \\ f_\tau(\tau) \end{bmatrix}, \quad [H(\tau)] = \begin{bmatrix} 0 & I \\ D & 0 \end{bmatrix}, \quad [D] = -\left[\sum_{i=1}^{N} M_{ij} + \varepsilon\Theta \cos\Theta\tau \left(\sum_{i=1}^{N} P_{ij}\right)\right].$$

From equation (32), Bolotin's method is used to find the regions of unstable solutions. As a first approximation, the periodic solutions have period $2T$ with $T = 2\pi/\omega$; thus these boundaries must be represented in Fourier series form. One has coefficients of $\sin(\bar{\Theta}\tau/2)$ as

$$-(\bar{\Theta}^2/4)\{a_1\}[I] + [M]\{a_1\} - \tfrac{1}{2}\varepsilon\bar{\Theta}^2\{a_1\}[P] = 0, \tag{35}$$

and coefficients of $\cos(\bar{\Theta}\tau/2)$ as

$$-(\bar{\Theta}^2/4)\{a_2\}[I] + [M]\{a_2\} + \tfrac{1}{2}\varepsilon\bar{\Theta}^2\{a_2\}[P] = 0. \quad (36)$$

The principal regions of instability are obtained from the zeros of the central elements of the system. Thus, one obtains

$$\begin{bmatrix} -(\bar{\Theta}^2/4)[I] + [M] - \tfrac{1}{2}\varepsilon\bar{\Theta}^2[P] & 0 \\ 0 & -\bar{\Theta}^2/4[I] + [M] + \tfrac{1}{2}\varepsilon\bar{\Theta}^2[P] \end{bmatrix} = 0. \quad (37)$$

Using Bolotin's method, two sets of homogeneous algebraic equations are obtained. These regions of instability are solved from the zeros of the central matrix elements of equations (33) and (37).

6. NUMERICAL RESULTS

The stability analyses using both Hsu's and Bolotin's methods are illustrated in Figure 2 for the system with a constant angular speed $\dot{\theta} = 1074{\cdot}61$ rad/s. The instability regions given by Hsu's method in the three cases of section 4.1 are shown in Figure 2(a) for ω_{ie} being near $\frac{1}{2}(\bar{\Theta} - \Theta)$, $i = 1, 2, \ldots, 4$; in Figure 2(b)

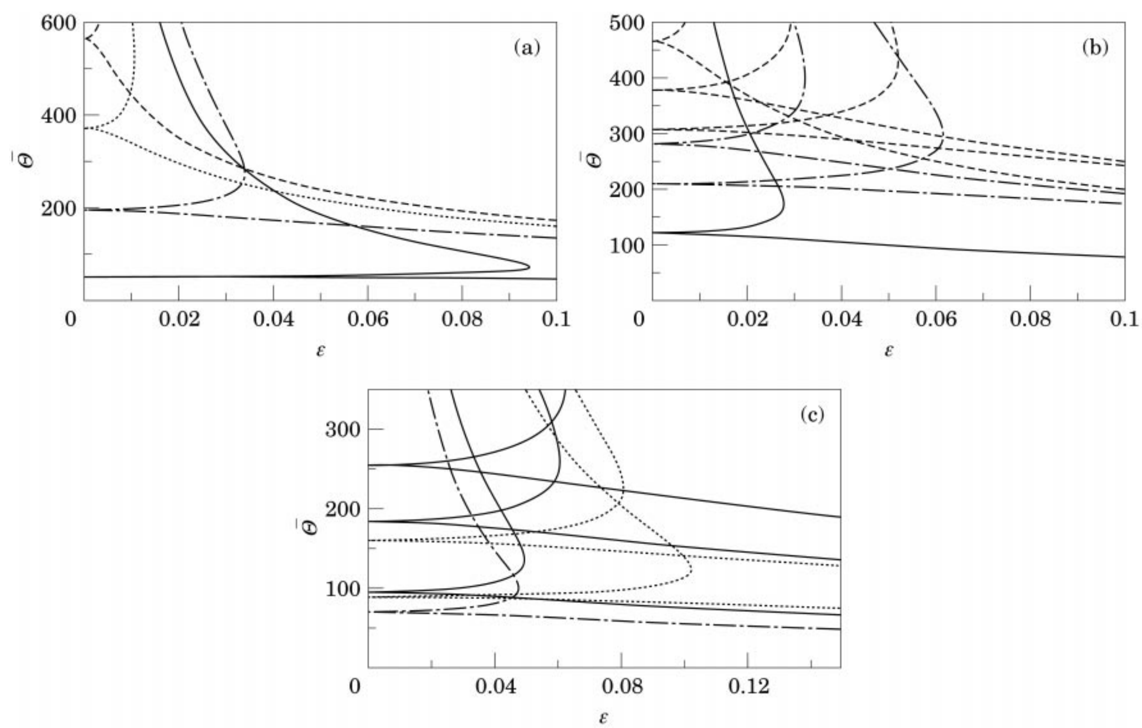

Figure 2. Instability regions with the constant angular velocity $\dot{\theta} = 1074{\cdot}61$ rad/s of a cantilever beam with a tip mass by using Hsu's method. (a) $\frac{1}{2}(\bar{\Theta} - \Theta) \cong \omega_{ie}$, $i = 1$ (—), $i = 2$ (– · – ·), $i = 3$ (· · ·) and $i = 4$ (- - -), (b) $\bar{\Theta} - \Theta \cong \omega_{ie} + \omega_{je}$, $\omega_{1e} + \omega_{2e}$ (—), $\omega_{1e} + \omega_{3e}$ (– · – ·), $\omega_{2e} + \omega_{3e}$ (– · – ·), $\omega_{1e} + \omega_{4e}$ (- - -), $\omega_{2e} + \omega_{4e}$ (- - -) and $\omega_{3e} + \omega_{4e}$ (- - -), (c) $\bar{\Theta} - \Theta \cong \omega_{ie} - \omega_{je}$, $\omega_{2e} - \omega_{1e}$ (– · – ·), $\omega_{3e} - \omega_{1e}$ (· · ·), $\omega_{3e} - \omega_{2e}$ (· · ·), $\omega_{4e} - \omega_{1e}$ (—), $\omega_{4e} - \omega_{2e}$ (—) and $\omega_{4e} - \omega_{3e}$ (—).

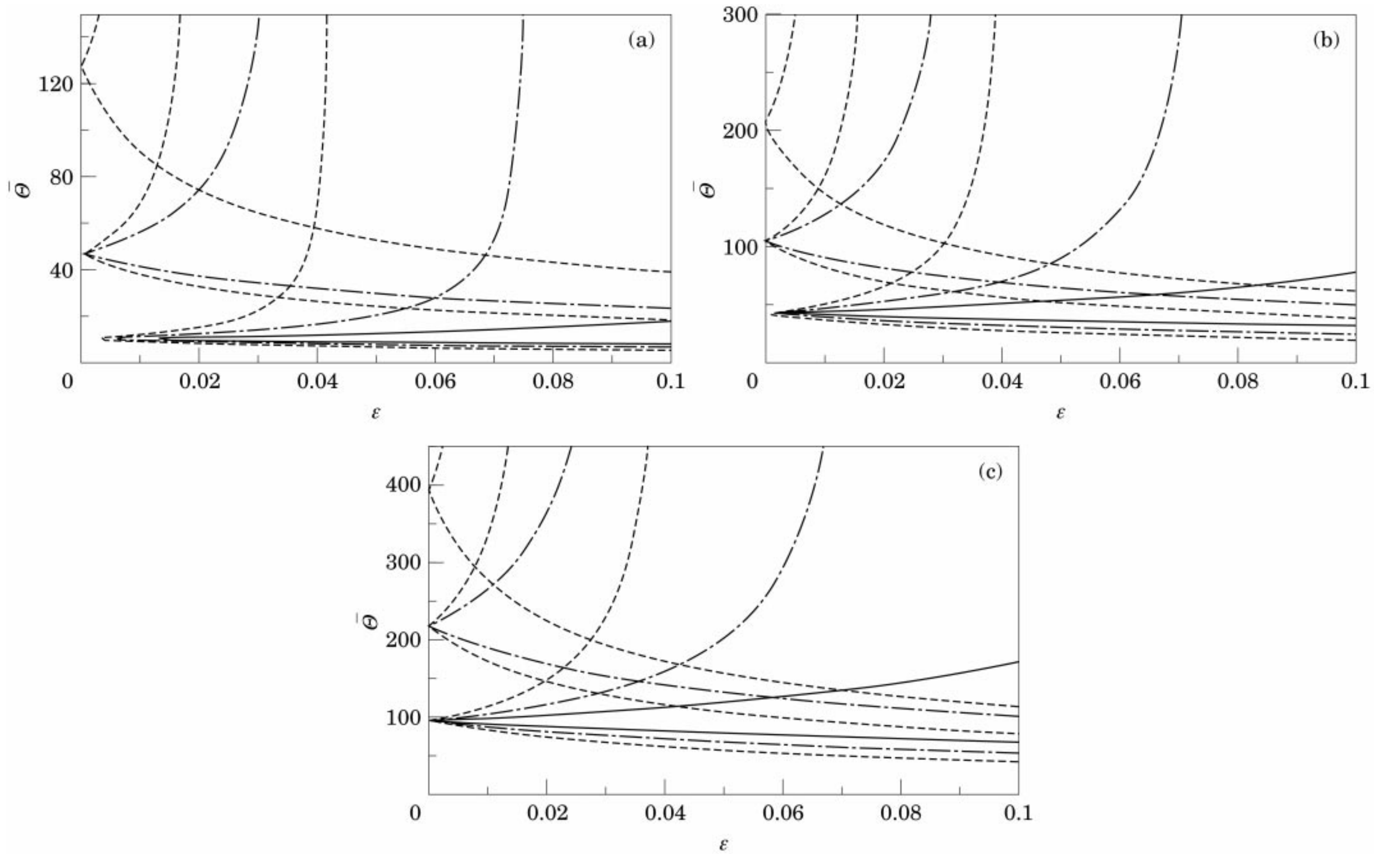

Figure 3. First three principal instability regions for different values of angle velocity. The principal matrices order 1 * 1 (—), 2 * 2 (– · – ·) and 3 * 3 (- - -). (a) $\dot{\theta}$ = 78·41 rad/s, (b) $\dot{\theta}$ = 474·41 rad/s, (c) $\dot{\theta}$ = 1074·61 rad/s.

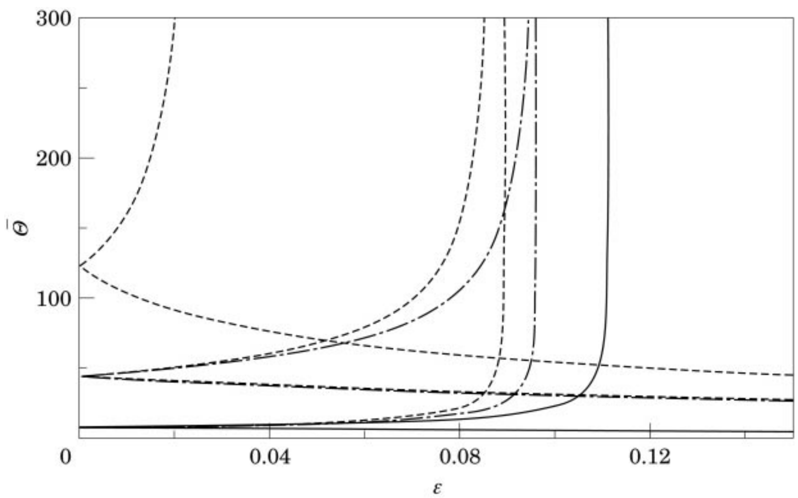

Figure 4. Instability boundaries of the beam without the tip mass and the base has only the axial translation. First (—), second (–·–·) and third (---) approximations obtained by equation (39).

combination resonance of the sum type for $\omega_{ie} + \omega_{je}$ being near $\bar{\Theta} - \Theta$; in Figure 2(c) combination resonance of the difference type for $\omega_{je} + \omega_{ie}$ being near $\bar{\Theta} - \Theta$.

Instability regions for different angular velocities of a cantilever beam attached to a translational/rotational base are shown in Figure 3(a–c). The results are obtained from equation (33) for the first three principal regions in the order of $1*1$ (—), $2*2$ (–·–·), $3*3$ (---). A higher value of the order corresponds to a larger instability region. The case of the beam is without tip

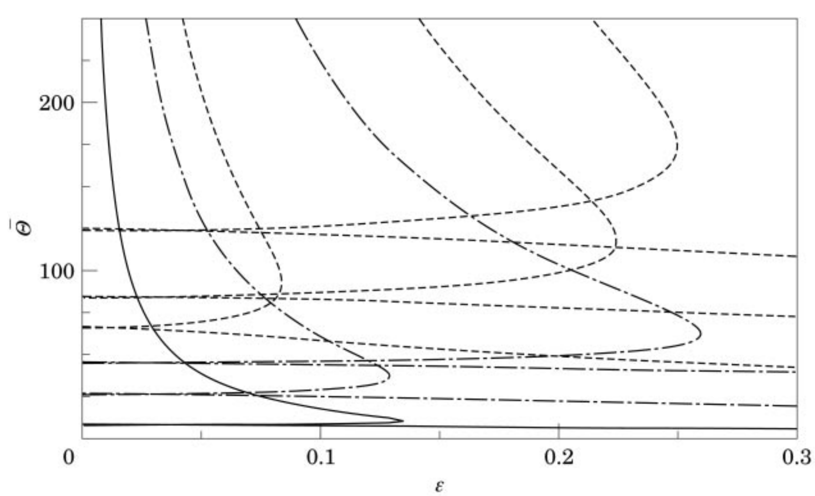

Figure 5. Instability regions for $\bar{\Theta} \cong (\omega_{ie} + \omega_{je})$ for $i, j = 1$ (—), 2 (–·–·), 3 (---) of the beam with the same parameters as in Figure 4 by using Hsu's method.

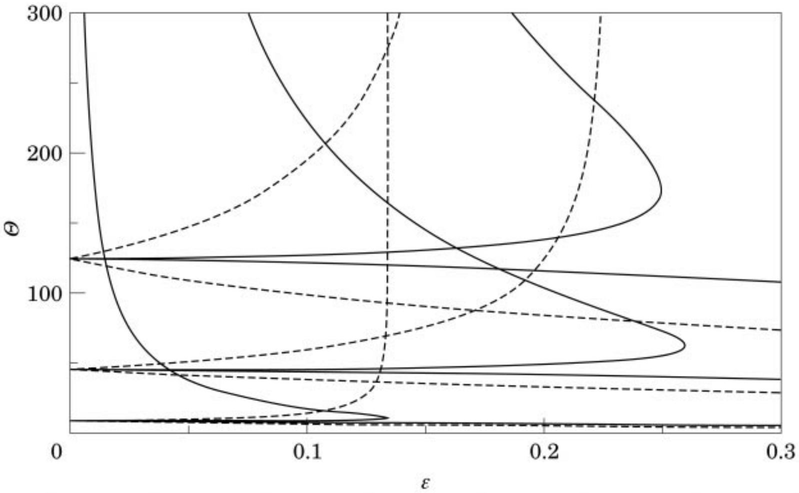

Figure 6. Instability regions compared by using Hsu's (—) and Bolotin's (- - -) methods for the beam without tip mass and the base with axial translation only.

mass and the base only has the axial translations. The first principal instability regions $\omega_{ip} + \omega_{jp}$, i, $j = 1$, 2, 3 taking equation (37) in the order of $1 * 1$ (—), $2 * 2$ (–––) and $3 * 3$ (- - -) matrices are shown in Figure 4. It is seen that the instability region enlarges as the order of the matrix increases. In Figure 5, the instability regions with the same parameters in Figure 4 are obtained by using Hsu's method, $2\omega_{1p}$ (—), $\omega_{1p} + \omega_{2p}$ (–·–·), $2\omega_{2p}$ (–·–·), $\omega_{1p} + \omega_{3p}$ (- - -), $\omega_{2p} + \omega_{3p}$ (- - -) and $2\omega_{3p}$ (- - -).

In Figure 6, the system has only the axial translation, and tip mass m_e is zero. The instability regions obtained by using Hsu's (—) and Bolotin's (–·–·)

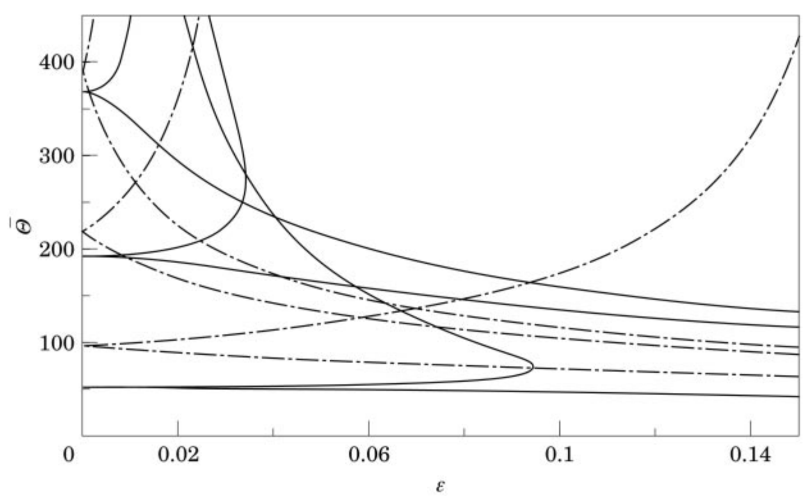

Figure 7. Instability regions compared by using Hsu's (—) and Bolotin's (–·–·) methods for the beam with tip mass and $\dot{\theta} = 1074{\cdot}61$ rad/s.

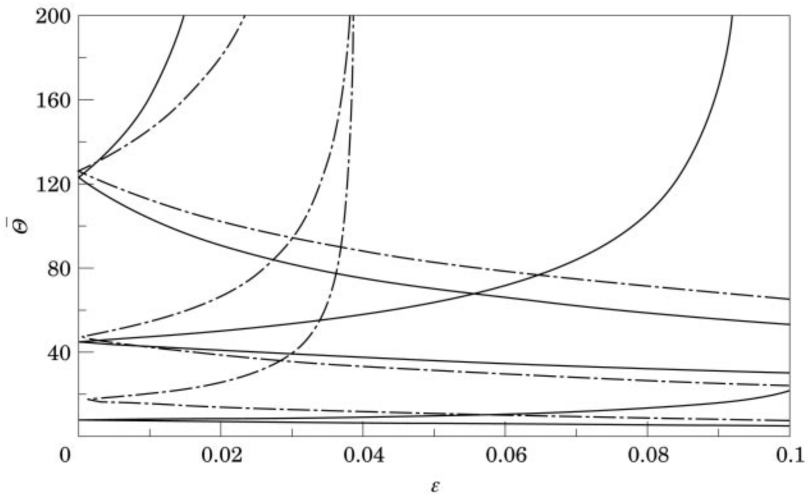

Figure 8. The effect of tip mass on the instability regions of the beam. With tip mass (– · – ·), without tip mass (—).

methods are compared for the principal matrices in the order of $1*1$, $2*2$ and $3*3$. Instability regions are compared in Figure 7 for the beam with the tip mass. It is observed that the results obtained by using Hsu's method (—) are not in agreement with those by using Bolotin's method (– · – ·). The main reason is that non-homogeneous terms are neglected in the first order approximations of Bolotin's method, and the principal matrices are only in the order of $1*1$, $2*2$ and $3*3$.

The effect of tip mass on the instability regions is shown in Figure 8. The instability regions for the cantilever beam with axial translations are obtained by using Bolotin's method in the order of the principal matrices $1*1$, $2*2$ and $3*3$. The presence of tip mass cuts off part of the instability regions, which borders on the frequency axis, and renders impossible the onset of resonance for sufficiently small coefficients of excitation.

7. CONCLUSIONS

The governing equations of a cantilever beam attached to a translation/rotational base are derived and reduced into the simple flexible beam model for dynamic stability analysis. The velocity and acceleration of the translational motion of the base are included in the formulation. In order to apply the Galerkin method, variable transformation is necessary to make the boundary conditions homogeneous. Periodic solutions composed of sine and cosine harmonics are used in Bolotin's method. The instability conditions are obtained from the zeros of the central elements of the system. Hsu's method is extended successfully to investigate the instability regions of the nonhomogeneous solutions. The effects of rotational angular speed and tip mass on the instability regions are investigated and compared by using both Hsu's and Bolotin's methods.

ACKNOWLEDGMENT

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APPENDIX A

A.1. TIMOSHENKO BEAM THEORY

A position vector of any material point $P(x, y)$ before deformation is

$$\mathbf{r}(x, y, t) = [a(t) + x\cos\theta - y\sin\theta]\mathbf{I} + [b(t) + x\sin\theta + y\cos\theta]\mathbf{J}, \tag{A1}$$

where **I**, **J** are the unit vectors of the fixed co-ordinate; $a(t)$ and $b(t)$ are time dependent.

The displacement field is

$$\mathbf{U}(x, y, t) = [(u - y\psi)\cos\theta - v\sin\theta]\mathbf{I} + [(u - y\psi)\sin\theta + v\cos\theta]\mathbf{J}, \tag{A2}$$

where $u(x, t)$ and $v(x, t)$ represent the axial and transverse displacements of the beam respectively; $\psi(x, t)$ is the rotatory angle of cross section due to bending alone.

Accordingly, the position vector of that point after deformation is

$$\mathbf{R}(x, y, t) = \mathbf{r}(x, y, t) + \mathbf{U}(x, y, t). \tag{A3}$$

Taking total derivative of $\mathbf{R}(x, y, t)$ with respect to time, one obtains

$$\begin{aligned}\dot{\mathbf{R}}(x, y, t) = {} & \{a_t + [(u_t - y\psi_t) - \dot{\theta}(y + v)]\cos\theta + [-v_t - \dot{\theta}(x + u - y\psi)]\sin\theta\}\mathbf{i} \\ & + \{b_t + [v_t + \dot{\theta}(x + u - y\psi)]\cos\theta + [(u_t - y\psi_t) - \dot{\theta}(y + v)]\sin\theta\}\mathbf{J}.\end{aligned} \tag{A4}$$

Therefore, the kinetic energy of the beam is

$$K.E. = \tfrac{1}{2}\rho\int_V (\dot{\mathbf{R}}\cdot\dot{\mathbf{R}})\,\mathrm{d}V = \int_0^\ell T\,\mathrm{d}x,$$

$$\begin{aligned}T = {} & \tfrac{1}{2}\rho A\langle\{a_t + [(u_t - y\psi_t) - \dot{\theta}(y + v)]\cos\theta + [-v_t - \dot{\theta}(x + u - y\psi)]\sin\theta\}^2 \\ & + \{b_t + [v_t + \dot{\theta}(x + u - y\psi)]\cos\theta + [(u_t - y\psi_t) - \dot{\theta}(y + v)]\sin\theta\}^2\rangle,\end{aligned} \tag{A5}$$

and of the tip mass is

$$\begin{aligned}T_m &= \tfrac{1}{2}m_e(\dot{\mathbf{R}}\cdot\dot{\mathbf{R}})|_{x=\ell} \\ &= \tfrac{1}{2}m_e\langle\{a_t + [(u_t(\ell, t) - y\psi_t(\ell, t)) - \dot{\theta}(y + v(\ell, t))]\cos\theta \\ &\quad + [-v_t(\ell, t) - \dot{\theta}(x + u(\ell, t) - y\psi(\ell, t))]\sin\theta\}^2 \\ &\quad + \{b_t + [v_t(\ell, t) + \dot{\theta}(x + u(\ell, t) - y\psi(\ell, t))]\cos\theta \\ &\quad + [(u_t(\ell, t) - y\psi_t(\ell, t)) - \dot{\theta}(y + v(\ell, t))]\sin\theta\}^2\rangle,\end{aligned} \tag{A6}$$

where ρ is mass density of the beam and m_e is the tip mass.

The Lagrangian strains in the corresponding directions are

$$\varepsilon_{xx} = u_x - y\psi_x + \tfrac{1}{2}v_x^2, \quad \varepsilon_{xy} = \tfrac{1}{2}(-\psi + v_x), \quad \varepsilon_{yy} = 0, \tag{A7}$$

where the higher order terms $\frac{1}{2}(u_x - y\psi_x)^2$ in ε_{xx}, $u_x\psi$ and $y\psi\psi_x$ in ε_{xy} and $\frac{1}{2}\psi^2$ in ε_{yy} are neglected. Hence, the total strain energy can be written as

$$S.E. = \tfrac{1}{2}\int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yy}\varepsilon_{yy})\,\mathrm{d}V = \int_0^\ell U^*\,\mathrm{d}x, \tag{A8}$$

where

$$U^* = \tfrac{1}{2}[EA(u_x + \tfrac{1}{2}v_x^2)^2 + EI\psi_x^2 + KGA(v_x - \psi)^2], \tag{A9}$$

and E is Young's modulus of the material. Therefore, the Lagrangian density of the system is

$$\mathcal{L}(x, t;\ u, u_x, u_t, v, v_x, v_t, \psi, \psi_x, \psi_t) = T - U^*. \tag{A10}$$

Hamilton's principle for the system is

$$\int_{t_1}^{t_2}\left[\int_0^{\ell}\delta\mathcal{L}\ \mathrm{d}x + \delta T_m\right]\mathrm{d}t = 0, \tag{A11}$$

where the variation of kinetic energy of the tip mass is

$$\begin{aligned}\int_{t_1}^{t_2}\delta T_m &= \int_{t_1}^{t_2} m_e\dot{\mathbf{R}}(\ell, 0, t)\cdot\delta\dot{\mathbf{R}}(\ell, 0, t)\ \mathrm{d}t\\ &= [m_e\dot{\mathbf{R}}(\ell, 0, t)\cdot\delta\mathbf{R}(\ell, 0, t)]_{t_1}^{t_2} - \int_{t_1}^{t_2}[m_e\ddot{\mathbf{R}}(\ell, 0, t)\cdot\delta\mathbf{R}(\ell, 0, t)]\ \mathrm{d}t. \end{aligned}\tag{A12}$$

Taking variation, applying the technique of integration by parts, substituting (A10) and (A12) into (A11) and collecting the like terms, one obtains

$$\begin{aligned}0 = &\int_{t_1}^{t_2}\int_0^{\ell}\left[\left(\frac{\partial\mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{u}} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial u_x}\right)\delta u + \left(\frac{\partial\mathcal{L}}{\partial v} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{v}} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial v_x}\right)\delta v\right.\\ &\left. + \left(\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\psi_x}\right)\delta\psi\right]\mathrm{d}x\ \mathrm{d}t\\ &+ \int_{t_1}^{t_2}\left[\left(\frac{\partial\mathcal{L}}{\partial u_x}\delta u + \frac{\partial\mathcal{L}}{\partial v_x}\delta v + \frac{\partial\mathcal{L}}{\partial\psi_x}\delta\psi\right)_0^{\ell} - m_e\ddot{\mathbf{R}}(\ell, 0, t)\cdot\delta\mathbf{R}(\ell, 0, t)\right]\mathrm{d}t. \end{aligned}\tag{A13}$$

After substituting equations (A5), (A6) and (A9) into (A13), one has the following governing equations

$$u: EA(u_{xx} + v_x v_{xx}) - \rho A u_{tt} + \rho A[\dot{\theta}^2(x + u) + 2\dot{\theta}v_t + \ddot{\theta}v - a_{tt}\cos\theta - b_{tt}\sin\theta] = 0, \tag{A14}$$

$$\begin{aligned}v: &EA(u_{xx}v_x + u_x v_{xx} + \tfrac{3}{2}v_x^2 v_{xx}) + KGA(v_{xx} - \psi_x) - \rho A v_{tt}\\ &+ \rho A[\dot{\theta}^2 v - 2\dot{\theta}u_t - \ddot{\theta}(x + u) - b_{tt}\cos\theta + a_{tt}\sin\theta] = 0, \end{aligned}\tag{A15}$$

$$\psi: EI\psi_{xx} + KGA(v_x - \psi) - \rho I\psi_{tt} + \rho I(\dot{\theta}^2\psi - \ddot{\theta}) = 0, \tag{A16}$$

and the associated boundary conditions

$$u(0, t) = v(0, t) = \psi(0, t) = \psi_x(\ell, t) = 0, \tag{A17a–d}$$

$$EA[u_x(\ell, t) + \tfrac{1}{2}v_x^2(\ell, t)] + m_e\{a_{tt}\cos\theta + b_{tt}\sin\theta + [u_{tt}(\ell, t) - \ddot{\theta}v(\ell, t) - 2\dot{\theta}v_t(\ell, t) - \dot{\theta}^2(\ell + u(\ell, t))]\} = 0, \quad \text{(A18)}$$

$$EA[u_x(\ell, t) + \tfrac{1}{2}v_x^2(\ell, t)]v_x(\ell, t) + KGA[v_x(\ell, t) - \psi(\ell, t)] + m_e\{b_{tt}\cos\theta - a_{tt}\sin\theta + [v_{tt}(\ell, t) + \ddot{\theta}(\ell + u(\ell, t)) + 2\dot{\theta}u_t(\ell, t) - \dot{\theta}^2 v(\ell, t)]\} = 0. \quad \text{(A19)}$$

A.2. EULER BEAM THEORY

If the beam is slender, Euler beam theory can be used to describe the beam system by setting $\psi = v_x$ and neglecting the shear deformation and rotary moment of inertia. The governing equations become

$$u : EA(u_{xx} + v_x v_{xx}) - \rho A u_{tt} + \rho A[\dot{\theta}^2(x + u) + 2\dot{\theta}v_t + \ddot{\theta}v - a_{tt}\cos\theta - b_{tt}\sin\theta] = 0, \quad \text{(A20)}$$

$$v : EA(u_{xx}v_x + u_x v_{xx} + \tfrac{3}{2}v_x^2 v_{xx}) - \rho A v_{tt} - EIv_{xxxx} + \rho A[\dot{\theta}^2 v - 2\dot{\theta}u_t - \ddot{\theta}(x + u) - b_{tt}\cos\theta + a_{tt}\sin\theta] = 0, \quad \text{(A21)}$$

and boundary conditions are

$$u(0, t) = v(0, t) = v_x(0, t) = v_{xx}(\ell, t) = 0, \quad \text{(A22a–d)}$$

$$EA[u_x(\ell, t) + \tfrac{1}{2}v_x^2(\ell, t)] + m_e\{a_{tt}\cos\theta + b_{tt}\sin\theta + [u_{tt}(\ell, t) - \ddot{\theta}v(\ell, t) - 2\dot{\theta}v_t(\ell, t) - \dot{\theta}^2(\ell + u(\ell, t))]\} = 0, \quad \text{(A23)}$$

$$EA[u_x(\ell, t) + \tfrac{1}{2}v_x^2(\ell, t)]v_x(\ell, t) - EIv_{xxx}(\ell, t) + m_e\{b_{tt}\cos\theta - a_{tt}\sin\theta + [v_{tt}(\ell, t) + \ddot{\theta}(\ell + u(\ell, t)) + 2\dot{\theta}u_t(\ell, t) - \dot{\theta}^2 v(\ell, t)]\} = 0. \quad \text{(A24)}$$

A.3. SIMPLE FLEXURE MODEL

In the simple flexible mode, the axial displacement $u(x, t)$ will be eliminated but retain the inertia effect of the translational and rotational motions of the base. Thus, one may define

$$p(x, t) = EA(u_x + \tfrac{1}{2}v_x^2), \quad p(\ell, t) = EA[u_x(\ell, t) + \tfrac{1}{2}v_x^2(\ell, t)]. \quad \text{(A25, A26)}$$

Neglecting u and u_{tt} in equation (A20), one gets

$$p_x(x, t) = -\rho A(\dot{\theta}^2 x + 2\dot{\theta} v_t + \ddot{\theta} v - a_{tt}\cos\theta - b_{tt}\sin\theta). \tag{A27}$$

Sequentially, substituting equation (A27) into (A21) and neglecting u and u_t, one has

$$[pv_x]_x - \rho A v_{tt} - EI v_{xxxx} + \rho A(\dot{\theta}^2 v - \ddot{\theta} x - b_{tt}\cos\theta + a_{tt}\sin\theta) = 0. \tag{A28}$$

From equations (A25) and (A26), one has

$$\begin{aligned} p(x, t) &= p(\ell, t) - \int_x^\ell \frac{\partial}{\partial x} p(x, t)\,\mathrm{d}x \\ &= -m_e\{a_{tt}\cos\theta + b_{tt}\sin\theta + [-\ddot{\theta} v(\ell, t) - 2\dot{\theta} v_t(\ell, t) - \dot{\theta}^2 \ell]\} \\ &\quad + \rho A[\dot{\theta}^2 \tfrac{1}{2}(\ell^2 - x^2) + (\ell - x)(-a_{tt}\cos\theta - b_{tt}\sin\theta) + \int_x^\ell (2\dot{\theta} v_t + \ddot{\theta} v)\,\mathrm{d}x]. \end{aligned} \tag{A29}$$

The governing equation (A28) is simplified as equation (1), and the boundary conditions (A22b–d) and (A24) become equations (2a–d).

APPENDIX B

$$D_{ij} = \int_0^1 \phi_i(\xi)\phi(\xi)\,\mathrm{d}\xi, \quad M_{ij} = \int_0^1 \phi_t^{(4)}(\xi)\phi_j(\xi)\,\mathrm{d}\xi, \quad N_{ij} = \int_0^1 \phi_i''(\xi)\phi_j(\xi)\,\mathrm{d}\xi,$$

$$Q_{ij} = \int_0^1 \phi_i'(\xi)\phi_j(\xi)\,\mathrm{d}\xi, \quad R_{ij} = \int_0^1 \xi\phi_i''(\xi)\phi_j(\xi)\,\mathrm{d}\xi, \quad T_{ij} = U_{ij} = k_{ij}A_j,$$

$$V_{ij} = k_{ij}B_j, \quad W_{ij} = -k_{ij}B_j, \quad E_{ij} = -k_{ij}z/\omega_j^2, \quad F_{ij} = \bar{\Theta}^2.$$

APPENDIX C

Here, Hsu's method is extended to investigate the non-homogeneous solutions of equation (14). As $\varepsilon = 0$, the solution of equation (14) is

$$f_i = A_{iq}\cos\omega_{ie}\tau + B_{iq}\sin\omega_{ie}\tau + z/\omega_{ie}^2. \tag{C1}$$

Equation (14) can be written in the first order form:

$$\mathrm{d}f_i/\mathrm{d}\tau = F_i, \tag{C2}$$

$$\mathrm{d}F_i/\mathrm{d}\tau + \omega_{ie}^2 f_i = -\varepsilon\bar{\Theta}^2\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n} Q_{ij} - \bar{m}_e \sum_{j=1}^{n} N_{ij}\right)\cos(\bar{\Theta}-\Theta)\tau\right] f_i + \varepsilon\bar{\Theta}^2 \sin(\bar{\Theta}-\Theta)\tau + z. \tag{C3}$$

A proposed solution for the first order analysis is of the form:

$$f_i = A_{iq}(\tau)\cos\omega_{ie}\tau + B_{iq}(\tau)\sin\omega_{ie}\tau + z/\omega_{ie}^2 + \varepsilon f_{ip}^{(1)}(\tau). \tag{C4}$$

The first three terms on the right-side are called the "variational" part of the solution, and the fourth term is the "perturbational" part of the solution. Further details will be based upon the two-mode approximation. These are the coefficients of ε^0,

$$(\mathrm{d}A_{1q}/\mathrm{d}\tau)\cos\omega_{1e}\tau + (\mathrm{d}B_{1q}/\mathrm{d}\tau)\sin\omega_{1e}\tau = 0, \tag{C5}$$

$$(\mathrm{d}A_{2q}/\mathrm{d}\tau)\cos\omega_{2e}\tau + (\mathrm{d}B_{2q}/\mathrm{d}\tau)\sin\omega_{2e}\tau = 0, \tag{C6}$$

$$-\omega_{1e}(\mathrm{d}A_{1q}/\mathrm{d}\tau)\sin\omega_{1e}\tau + \omega_{1e}(\mathrm{d}B_{1q}/\mathrm{d}\tau)\cos\omega_{1e}\tau = -z/\omega_{1e}^2 + z, \tag{C7}$$

$$-\omega_{2e}(\mathrm{d}A_{2q}/\mathrm{d}\tau)\sin\omega_{2e}\tau + \omega_{2e}(\mathrm{d}B_{2q}/\mathrm{d}\tau)\cos\omega_{2e}\tau = -z/\omega_{2e}^2 + z. \tag{C8}$$

The perturbational equations are obtained from the coefficients of ε^1,

$$\begin{aligned}\frac{\mathrm{d}f_1}{\mathrm{d}\tau} + \omega_{1e}^2 f_1 = -\tfrac{1}{2}\Bigg[&\sum_{j=1}^{2} T_{1j}\cos(\bar{\Theta}-\Theta+\omega_{ie})\tau + \sum_{j=1}^{2} U_{1j}\cos(\bar{\Theta}-\Theta-\omega_{ie})\tau \\ &+ \sum_{j=1}^{2} V_{1j}\sin(\bar{\Theta}-\Theta+\omega_{ie})\tau + \sum_{j=1}^{2} W_{1j}\sin(\bar{\Theta}-\Theta-\omega_{1e})\tau\Bigg] \\ &- 2\sum_{j=1}^{2} E_{1j}\cos(\bar{\Theta}-\Theta)\tau - 2\sum_{j=1}^{2} F_{1j}\sin(\bar{\Theta}-\Theta)\tau,\end{aligned} \tag{C9}$$

$$\begin{aligned}\frac{\mathrm{d}f_2}{\mathrm{d}\tau} + \omega_{2e}^2 f_2 = -\tfrac{1}{2}\Bigg[&\sum_{j=1}^{2} T_{2j}\cos(\bar{\Theta}-\Theta+\omega_{ie})\tau + \sum_{j=1}^{2} U_{2j}\cos(\bar{\Theta}-\Theta-\omega_{ie})\tau \\ &+ \sum_{j=1}^{2} V_{2j}\sin(\bar{\Theta}-\Theta+\omega_{ie})\tau + \sum_{j=1}^{2} W_{2j}\sin(\bar{\Theta}-\Theta-\omega_{2e})\tau\Bigg] \\ &- 2\sum_{j=1}^{2} E_{2j}\cos(\bar{\Theta}-\Theta)\tau - 2\sum_{j=1}^{2} F_{2j}\sin(\bar{\Theta}-\Theta)\tau.\end{aligned} \tag{C10}$$

The particular integrals to the perturbational equations by using Hsu's method are shown in equation (16). The non-homogeneous equation is stable (perturbation part) in this case; the denominator $\omega_{1e}^2 - (\bar{\Theta} - \Theta)^2$ in equation (16) is nearly equal to zero. A solution is found for $\omega_{1e} \approx \bar{\Theta} - \Theta$ and described by $\omega_{1e} + \varepsilon\lambda = \bar{\Theta} - \Theta$, where λ is a finite real number.

The offending terms are then associated with the first perturbational equation (C9), and the perturbational equations after removing the offending terms are

$$\frac{\mathrm{d}^2 f_1}{\mathrm{d}\tau^2} + \omega_{1e}^2 f_1 = -\tfrac{1}{2}\Bigg[\sum_{j=1}^{2} T_{1j}\cos(\bar{\Theta} - \Theta + \omega_{je})\tau + V_{1j}\sin(\bar{\Theta} - \Theta + \omega_{je})\tau$$
$$+ \sum_{j=1}^{2} U_{1j}\cos(\bar{\Theta} - \Theta - \omega_{je})\tau + W_{1j}\sin(\bar{\Theta} - \Theta - \omega_{je})\tau\Bigg], \quad \text{(C11)}$$

$$\frac{\mathrm{d}^2 f_2}{\mathrm{d}\tau^2} + \omega_{2e}^2 f_2 = -2[E_{2j}\cos(\bar{\Theta} - \Theta)\tau + F_{2j}\sin(\bar{\Theta} - \Theta)\tau]$$
$$-\tfrac{1}{2}\Bigg[\sum_{j=1}^{2} T_{2j}\cos(\bar{\Theta} - \Theta + \omega_{je})\tau + V_{2j}\sin(\bar{\Theta} - \Theta + \omega_{j})\tau$$
$$+ \sum_{j=1}^{2} U_{2j}\cos(\bar{\Theta} - \Theta - \omega_{je})\tau + W_{2j}\sin(\bar{\Theta} - \Theta - \omega_{je})\tau\Bigg]. \quad \text{(C12)}$$

The variational equations become

$$(\mathrm{d}A_{iq}/\mathrm{d}\tau)\cos\omega_{ie}\tau + (\mathrm{d}B_{iq}/\mathrm{d}\tau)\sin\omega_{ie}\tau = 0, \quad i = 1, 2, \quad \text{(C13)}$$

$$-\omega_{1e}(\mathrm{d}A_{1q}/\mathrm{d}\tau)\sin\omega_{1e}\tau + \omega_{1e}(\mathrm{d}B_{1q}/\mathrm{d}\tau)\cos\omega_{1e}\tau$$
$$= \varepsilon\{-2[E_{1j}\cos(\bar{\Theta} - \Theta)\tau + F_{1j}\sin(\bar{\Theta} - \Theta)\tau]\} - z/\omega_{1e}^2 + z, \quad \text{(C14)}$$

$$-\omega_{2e}(\mathrm{d}A_{2q}/\mathrm{d}\tau)\sin\omega_{2e}\tau + \omega_{2e}(\mathrm{d}B_{2q}/\mathrm{d}\tau)\cos\omega_{2e}\tau = -z/\omega_{2e}^2 + z. \quad \text{(C15)}$$

From equations (C13–15), one finds

$$\mathrm{d}A_{1q}/\mathrm{d}\tau = -\mathrm{d}B_{1q}/\mathrm{d}\tau\tan\omega_{1e}\tau, \quad \mathrm{d}A_{2q}/\mathrm{d}\tau = -\mathrm{d}B_{2q}/\mathrm{d}\tau\tan\omega_{2e}\tau, \quad \text{(C16, C17)}$$

$$\mathrm{d}B_{1q}/\mathrm{d}\tau = -(2\varepsilon/\omega_{1e})[E_{ij}\cos(\omega_{1e}\tau)\cos(\bar{\Theta} - \Theta)\tau + F_{ij}\cos(\omega_{1e}\tau)\sin(\bar{\Theta} - \Theta)\tau]$$
$$-(1/\omega_{1e})(z/\omega_{1e}^2 - z)\cos\omega_{1e}\tau, \quad \text{(C18)}$$

$$\mathrm{d}B_{2q}/\mathrm{d}\tau = (1/\omega_{2q})(-z/\omega_{2q}^2 + z)\cos\omega_{2e}\tau. \quad \text{(C19)}$$

Substituting $\omega_{1e} = \bar{\Theta} - \Theta - \varepsilon\lambda$ into equation (C18), and applying trigonometric identities, one obtains

$$\mathrm{d}B_{1q}/\mathrm{d}\tau = -(\varepsilon/\omega_{1e})\langle E_{ij}\{\cos[2(\bar{\Theta}-\Theta)-\varepsilon\lambda]\tau+\cos(\varepsilon\lambda\tau)\} + F_{ij}\{\sin[2(\bar{\Theta}-\Theta)-\varepsilon\lambda]\tau+\sin(\varepsilon\lambda\tau)\}\rangle - (1/\omega_{1e})(z/\omega_{1e}^2 - z)\cos\omega_{1e}\tau. \tag{C20}$$

From equation (C20), B_{1q} can easily be found for $\varepsilon = 0$ or $\lambda = 0$ as

$$B_{1q} = -\varepsilon/\omega_{1e}E_{1j}\{\sin[2(\bar{\Theta}-\Theta)]\tau/2(\bar{\Theta}-\Theta)\} + (\varepsilon/\omega_{1e})F_{1j}\{\cos[2(\bar{\Theta}-\Theta)]\tau/2(\bar{\Theta}-\Theta)\} - (z/\omega_{1e})((1-\omega_{1e}^2)/\omega_{1e}^2)[\sin(\bar{\Theta}-\Theta)\tau/(\bar{\Theta}-\Theta)] + c, \tag{C21}$$

where c is a constant. When $\varepsilon \neq 0$ and $\lambda \neq 0$, (C21) is simplified as

$$B_{1q} = -(\varepsilon/\omega_{1e})E_{1j}\{\sin[2(\bar{\Theta}-\Theta)-\varepsilon\lambda]\tau/[2(\bar{\Theta}-\Theta-\varepsilon\lambda)] + \sin\varepsilon\lambda\tau/\varepsilon\lambda\} + \frac{\varepsilon}{\omega_{1e}}F_{1j}\left\{\frac{\cos[2(\bar{\Theta}-\Theta)-\varepsilon\lambda]\tau}{2(\bar{\Theta}-\Theta)-\varepsilon\lambda} + \frac{\cos\varepsilon\lambda\tau}{\varepsilon\lambda}\right\} - \frac{z}{\omega_{1e}}\left(\frac{1-\omega_1^2}{\omega_1^2}\right)\left[\frac{\sin(\bar{\Theta}-\Theta-\varepsilon\lambda)\tau}{\bar{\Theta}-\Theta-\varepsilon\lambda}\right] + c. \tag{C22}$$

From (C16) and (C20), one also has

$$A_{1q} = (\varepsilon/\omega_{1e})E_{1j}\{-\cos[2(\bar{\Theta}-\Theta)-\varepsilon\lambda]\tau/[2(\bar{\Theta}-\Theta)-\varepsilon\lambda] + \cos(\varepsilon\lambda\tau)/\varepsilon\lambda\} + \frac{\varepsilon}{\omega_{1e}}F_{1j}\left\{-\frac{\sin[2(\bar{\Theta}-\Theta)-\varepsilon\lambda]\tau}{2(\bar{\Theta}-\Theta)-\varepsilon\lambda} + \frac{\sin(\varepsilon\lambda\tau)}{\varepsilon\lambda}\right\} - \frac{z}{\omega_{1e}}\left(\frac{1-\omega_{1e}^2}{\omega_{1e}^2}\right)\left[\frac{\cos\omega_{1e}\tau}{\bar{\Theta}-\Theta-\varepsilon\lambda}\right] + c_1, \tag{C23}$$

where c_1 is a constant.